1 Topological and metric spaces

1.1 Basic Definitions

Definition 1.1 (Topology)**.** Let S be a set. A subset $\mathcal{T}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *topology* iff it has the following properties:

- $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$.
- Let $\{U_i\}_{i\in I}$ be a family of elements in $\mathcal{T}$. Then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
- Let $U, V \in \mathcal{T}$. Then $U \cap V \in \mathcal{T}$.

A set equipped with a topology is called a *topological space*. The elements of $\mathcal{T}$ are called the *open* sets in S. A complement of an open set in S is called a *closed* set.

Definition 1.2. Let S be a topological space and $x \in S$. Then a subset $U \subseteq S$ is called a *neighborhood* of x iff it contains an open set which in turn contains x.

Definition 1.3. Let S be a topological space and U a subset. The *closure* $\overline{U}$ of U is the smallest closed set containing U. The *interior* $\mathring{U}$ of U is the largest open set contained in U. U is called *dense* in S iff $\overline{U} = S$.

Definition 1.4 (base)**.** Let $\mathcal{T}$ be a topology. A subset $\mathcal{B}$ of $\mathcal{T}$ is called a *base* of $\mathcal{T}$ iff the elements of $\mathcal{T}$ are precisely the unions of elements of $\mathcal{B}$. It is called a *subbase* iff the elements of $\mathcal{T}$ are precisely the finite intersections of unions of elements of $\mathcal{B}$.

Proposition 1.5. *Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. $\mathcal{B}$ is the base of a topology on S iff it satisfies all of the following properties:*

- $\emptyset \in \mathcal{B}$.
- *For every $x \in S$ there is a set $U \in \mathcal{B}$ such that $x \in U$.*
- *Let $U, V \in \mathcal{B}$. Then there exits a family $\{W_\alpha\}_{\alpha\in A}$ of elements of $\mathcal{B}$ such that $U \cap V = \bigcup_{\alpha\in A} W_\alpha$.*

Proof. **Exercise.** □

Definition 1.6. Let S be a topological space and p a point in S. We call a family $\{U_\alpha\}_{\alpha\in A}$ of open neighborhoods of p a *neighborhood base* at p iff for any neighborhood V of p there exists $\alpha \in A$ such that $U_\alpha \subseteq V$.

Definition 1.7 (Continuity)**.** Let S, T be topological spaces. A map $f : S \to T$ is called *continuous* iff for every open set $U \in T$ the preimage $f^{-1}(U)$ in S is open. We denote the space of continuous maps from S to T by $\mathrm{C}(S, T)$.

Proposition 1.8. *Let S, T, U be topological spaces, $f \in \mathrm{C}(S,T)$ and $g \in \mathrm{C}(T,U)$. Then, the composition $g \circ f : S \to U$ is continuous.*

Proof. Immediate. $\square$

Definition 1.9. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on the set S. Then, $\mathcal{T}_1$ is called *finer* than $\mathcal{T}_2$ and $\mathcal{T}_2$ is called *coarser* than $\mathcal{T}_1$ iff all open sets of $\mathcal{T}_2$ are also open sets of $\mathcal{T}_1$.

Definition 1.10 (Induced Topology)**.** Let S be a topological space and U a subset. Consider the topology given on U by the intersection of each open set on S with U. This is called the *induced topology* on U.

Exercise 1. Show that the induced topology is the coarsest topology on U making the inclusion $U \hookrightarrow S$ continuous.

Definition 1.11 (Product Topology)**.** Let S be the cartesian product $S = \prod_{\alpha \in I} S_\alpha$ of a family of topological spaces. Consider subsets of S of the form $\prod_{\alpha \in I} U_\alpha$ where finitely many U_α are open sets in S_α and the others coincide with the whole space $U_\alpha = S_\alpha$. These subsets form the base of a topology on S which is called the *product topology*.

Exercise 2. Show that alternatively, the product topology can be characterized as the coarsest topology on $S = \prod_{\alpha \in I} S_\alpha$ such that all projections $S \twoheadrightarrow S_\alpha$ are continuous.

Proposition 1.12. *Let S, T, X be topological spaces and $f \in \mathrm{C}(S \times T, X)$, where $S \times T$ carries the product topology. Then the map $f_x : T \to X$ defined by $f_x(y) = f(x,y)$ is continuous for every $x \in S$.*

Proof. Fix $x \in S$. Let U be an open set in X. We want to show that $W := f_x^{-1}(U)$ is open. We do this by finding for any $y \in W$ an open neighborhood of y contained in W. If W is empty we are done, hence assume that this is not so. Pick $y \in W$. Then $(x,y) \in f^{-1}(U)$ with $f^{-1}(U)$ open by continuity of f. Since $S \times T$ carries the product topology there must be open sets $V_x \subseteq S$ and $V_y \subseteq T$ with $x \in V_x$, $y \in V_y$ and $V_x \times V_y \subseteq f^{-1}(U)$. But clearly $V_y \subseteq W$ and we are done. $\square$

Definition 1.13 (Quotient Topology)**.** Let S be a topological space and $\sim$ an equivalence relation on S. Then, the *quotient topology* on $S/\sim$ is the finest topology such that the quotient map $S \twoheadrightarrow S/\sim$ is continuous.

1.2 Some properties of topological spaces

In a topological space it is useful if two distinct points can be distinguished by the topology. A strong form of this distinguishability is the *Hausdorff property*.

Definition 1.14 (Hausdorff)**.** Let S be a topological space. Assume that given any two distinct points $x, y \in S$ we can find open sets $U, V \subset S$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. Then, S is said to have the *Hausdorff property.* We also say that S is a *Hausdorff space.*

Definition 1.15. Let S be a topological space. S is called *first-countable* iff there exists a countable neighborhood base at each point of S. S is called *second-countable* iff the topology of S admits a countable base.

Definition 1.16 (open cover)**.** Let S be a topological space and $U \subseteq S$ a subset. A family of open sets $\{U_\alpha\}_{\alpha \in A}$ is called an *open cover* of U iff $U \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Proposition 1.17. *Let S be a second-countable topological space and $U \subseteq S$ a subset. Then, every open cover of U contains a countable subcover.*

Proof. **Exercise.** □

Definition 1.18. Let S be a topological space and $U \subseteq S$ a subset. U is called *compact* iff every open cover of U contains a finite subcover.

Proposition 1.19. *A closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed.*

Proof. **Exercise.** □

Proposition 1.20. *The image of a compact set under a continuous map is compact.*

Proof. **Exercise.** □

1.3 Sequences and convergence

Definition 1.21 (Convergence of sequences)**.** Let $x := \{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in a topological space S. We say that x has an *accumulation point (or limit point)* p iff for every neighborhood U of x we have $x_k \in U$ for infinitely many $k \in \mathbb{N}$. We say that x *converges* to a point p iff for any neighborhood U of p there is a number $n \in \mathbb{N}$ such that for all $k \geq n$: $x_k \in U$.

Proposition 1.22. *Let S, T be topological spaces and $f : S \to T$. If f is continuous, then for any $p \in S$ and sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to p, the sequence $f\{(x_n)\}_{n \in \mathbb{N}}$ in T converges to $f(p)$. Conversely, if S is first countable and for any $p \in S$ and sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to p, the sequence $f\{(x_n)\}_{n \in \mathbb{N}}$ in T converges to $f(p)$, then f is continuous.*

Proof. **Exercise.** □

Proposition 1.23. *Let S be Hausdorff space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in S which converges to a point $x \in S$. Then, $\{x_n\}_{n\in\mathbb{N}}$ does not converge to any other point in S.*

Proof. **Exercise.** □

Definition 1.24. Let S be a topological space and $U \subseteq S$ a subset. The set $\overline{U}^s$ of points to which sequences of elements of U converge is called the *sequential closure* of S.

Proposition 1.25. *Let S be a topological space and $U \subseteq S$ a closed subset. Let x be a sequence of points in U which has an accumulation point $p \in S$. Then, $p \in U$.*

Proof. Suppose $p \notin U$. Since U is closed $S \setminus U$ is an open neighborhood of p. But $S \setminus U$ does not contain any point of x, so p cannot be accumulation point of x. This is a contradiction. □

Corollary 1.26. *Let S be a topological space and U a subset. Then, $\overline{U}^s \subseteq \overline{U}$.*

Proof. Immediate. □

Proposition 1.27. *Let S be a first-countable topological space and U a subset. Then, $\overline{U}^s = \overline{U}$.*

Proof. **Exercise.** □

Definition 1.28. Let S be a topological space and $U \subseteq S$ a subset. U is said to be *limit point compact* iff every sequence in S has an accumulation point (limit point) in U. U is called *sequentially compact* iff every sequence of elements of U contains a subsequence converging to a point in U.

Proposition 1.29. *Let S be a first-countable topological space and $x = \{x_n\}_{n\in\mathbb{N}}$ a sequence in S with accumulation point p. Then, x has a subsequence that converges to p.*

Proof. By first-countability choose a countable neighborhood base $\{U_n\}_{n\in\mathbb{N}}$ at p. Now consider the family $\{W_n\}_{n\in\mathbb{N}}$ of open neighborhoods $W_n := \bigcap_{k=1}^n U_k$ at p. It is easy to see that this is again a countable neighborhood base at p. Moreover, it has the property that $W_n \subseteq W_m$ if $n \geq m$. Now, Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in W_1$. Recursively, choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in W_{k+1}$. This is possible since W_{k+1} contains infinitely many points of x. Let V be a neighborhood of p. There exists some $k \in \mathbb{N}$ such that $U_k \subseteq V$. By construction, then $W_m \subseteq W_k \subseteq U_k$ for all $m \geq k$ and hence $x_{n_m} \in V$ for all $m \geq k$. Thus, the subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$ converges to p. □

Corollary 1.30. *The notions of limit point compactness and sequential compactness coincide for first-countable spaces.*

Proof. Immediate. □

Proposition 1.31. *A compact set is limit point compact.*

Proof. Consider a sequence x in a compact set S. Suppose x does not have an accumulation point. Then, for each point $p \in S$ we can choose an open neighborhood U_p which contains only finitely many points of x. However, by compactness, S is covered by finitely many of the sets U_p. But their union can only contain a finite number of points of x, a contradiction. □

1.4 Metric and pseudometric spaces

Definition 1.32. Let S be a set and $d : S \times S \to \mathbb{R}_0^+$ a map with the following properties:

- $d(x, y) = d(y, x) \quad \forall x, y \in S$. (symmetry)
- $d(x, z) \le d(x, y) + d(y, z) \quad \forall x, y, z \in S$. (triangle inequality)
- $d(x, x) = 0 \quad \forall x \in S$.

Then d is called a *pseudometric* on S. S is also called a *pseudometric space.* Suppose d also satisfies

- $d(x, y) = 0 \implies x = y \quad \forall x, y \in S$. (definiteness)

Then d is called a *metric* on S and S is called a *metric space.*

Definition 1.33. Let S be a pseudometric space, $x \in S$ and $r > 0$. Then the set $B_r(x) := \{y \in S : d(x, y) < r\}$ is called the *open ball* of radius r centered around x in S. The set $\overline{B}_r(x) := \{y \in S : d(x, y) \le r\}$ is called the *closed ball* of radius r centered around x in S.

Proposition 1.34. *Let S be a pseudometric space. Then, the open balls in S together with the empty set form the basis of a topology on S. This topology is first-countable and such that closed balls are closed. Moreover, the topology is Hausdorff iff S is metric.*

Proof. **Exercise.** □

Definition 1.35. A topological space is called *(pseudo)metrizable* iff there exists a (pseudo)metric such that the open balls given by the (pseudo)metric are a basis of its topology.

Proposition 1.36. *In a pseudometric space any open ball can be obtained as the countable union of closed balls. Similarly, any closed ball can be obtained as the countable intersection of open balls.*

Proof. **Exercise.** □

Proposition 1.37. *Let S be a set equipped with two pseudometrics d^1 and d^2. Then, the topology generated by d^2 is finer than the topology generated by d^1 iff for all $x \in S$ and $r_1 > 0$ there exists $r_2 > 0$ such that $B^2_{r_2}(x) \subseteq B^1_{r_1}(x)$. In particular, d^1 and d^2 generate the same topology iff the condition holds both ways.*

Proof. **Exercise.** □

Proposition 1.38 (epsilon-delta criterion)**.** *Let S, T be pseudometric spaces and $f : S \to T$ a map. Then, f is continuous iff for every $x \in S$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.*

Proof. **Exercise.** □

1.5 Elementary properties of pseudometric spaces

Proposition 1.39. *Let S be a pseudometric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x converges to $p \in S$ iff for any $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0 : d(x_n, p) < \epsilon$.*

Proof. Immediate. □

Definition 1.40. Let S be a pseudometric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x is called a *Cauchy sequence* iff for all $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0 : d(x_n, x_m) < \epsilon$.

Exercise 3. Give an example of a set S, a sequence x in S and two metrics d^1 and d^2 on S that generate the same topology, but such that x is Cauchy with respect to d^1, but not with respect to d^2.

Proposition 1.41. *Any converging sequence in a pseudometric space is a Cauchy sequence.*

Proof. **Exercise.** □

Proposition 1.42. *Suppose x is a Cauchy sequence in a pseudometric space. If p is accumulation point of x then x converges to p.*

Proof. **Exercise.** □

Definition 1.43. Let S be a pseudometric space and $U \subseteq S$ a subset. If every Cauchy sequence in U converges to a point in U, then U is called *complete*.

Proposition 1.44. *A complete subset of a metric space is closed. A closed subset of a complete pseudometric space is complete.*

Proof. **Exercise.** □

Exercise 4. Give an example of a complete subset of a pseudometric space that is not closed.

Definition 1.45 (Totally boundedness)**.** Let S be a pseudometric space. A subset $U \subseteq S$ is called *totally bounded* iff for any $r > 0$ the set U admits a cover by finitely many open balls of radius r.

Proposition 1.46. *A subset of a pseudometric space is compact iff it is complete and totally bounded.*

Proof. We first show that compactness implies totally boundedness and completeness. Let U be a compact subset. Then, for $r > 0$ cover U by open balls of radius r centered at every point of U. Since U is compact, finitely many balls will cover it. Hence, U is totally bounded. Now, consider a Cauchy sequence x in U. Since U is compact x must have an accumulation point $p \in U$ (Proposition 1.31) and hence (Proposition 1.42) converge to p. Thus, U is complete.

We proceed to show that completeness together with totally boundedness imply compactness. Let U be a complete and totally bounded subset. Assume U is not compact and choose a covering $\{U_\alpha\}_{\alpha \in A}$ of U that does not admit a finite subcovering. On the other hand, U is totally bounded and admits a covering by finitely many open balls of radius $1/2$. Hence, there must be at least one such ball B_1 such that $C_1 := B_1 \cap U$ is not covered by finitely many U_α. Choose a point x_1 in C_1. Observe that C_1 itself is totally bounded. Inductively, cover C_n by finitely many open balls of radius $2^{-(n+1)}$. For at least one of those, call it B_{n+1}, $C_{n+1} := B_{n+1} \cap C_n$ is not covered by finitely many U_α. Choose a point x_{n+1} in C_{n+1}. This process yields a Cauchy sequence $x := \{x_k\}_{k \in \mathbb{N}}$. Since U is complete the sequence converges to a point $p \in U$. There must be $\alpha \in A$ such that $p \in U_\alpha$. Since U_α is open there exists $r > 0$ such that $B(p, r) \subseteq U_\alpha$. This implies, $C_n \subseteq U_\alpha$ for all $n \in \mathbb{N}$ such that $2^{-n+1} < r$. However, this is a contradiction to the C_n not being finitely covered. Hence, U must be compact. □

Proposition 1.47. *The notions of compactness, limit point compactness and sequential compactness are equivalent in a pseudometric space.*

Proof. **Exercise.** □

Proposition 1.48. *A totally bounded pseudometric space is second-countable.*

Proof. **Exercise.** □

Proposition 1.49. *Let S be equipped with a pseudometric d. Then $p \sim q \iff d(p, q) = 0$ for $p, q \in S$ defines an equivalence relation on S. The prescription $\tilde{d}([p], [q]) := d(p, q)$ for $p, q \in S$ is well defined and yields a metric $\tilde{d}$ on the quotient space $S/\sim$. The topology induced by this metric on $S/\sim$ is the quotient topology with respect to that induced by d on S. Moreover, $S/\sim$ is complete iff S is complete.*

Proof. **Exercise.** □

Exercise 5. Show the following universal property of the quotient construction given above: Let S be a pseudometric space, T be a Hausdorff space and $f : S \to T$ a continuous map. Then, there exists a unique continuous map $\tilde{f} : S/\sim \to T$ such that $f = \tilde{f} \circ q$, where q is the quotient map $S \to S/\sim$.

1.6 Completion of metric spaces

Often it is desirable to work with a complete metric space when one is only given a non-complete metric space. To this end one can construct the *completion* of a metric space. This is detailed in the following exercise.

Exercise 6. Let S be a metric space.

- Let $x := \{x_n\}_{n\in\mathbb{N}}$ and $y := \{y_n\}_{n\in\mathbb{N}}$ be Cauchy sequences in S. Show that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists.
- Let T be the set of Cauchy sequences in S. Define the function $\tilde{d} : T \times T \to \mathbb{R}_0^+$ by $\tilde{d}(x, y) := \lim_{n\to\infty} d(x_n, y_n)$. Show that $\tilde{d}$ defines a pseudometric on T.
- Define $\overline{S}$ as the metric quotient $T/\sim$ as in Proposition 1.49.
- Show that $\overline{S}$ is complete. [Hint: First show that given a Cauchy sequence x in S and a subsequence x' of x we have $\tilde{d}(x, x') = 0$. That is, $x \sim y$ in T. Use this to show that for any Cauchy sequence x in S an equivalent Cauchy Sequence x' can be constructed which has a specific asymptotic behavior. For example, x' can be made to satisfy $d(x'_n, x'_m) < \frac{1}{\min(m,n)}$. Now a Cauchy sequence $\hat{x} = \{\hat{x}^n\}_{n\in\mathbb{N}}$ in $\overline{S}$ consists of equivalence classes $\hat{x}^n$ of Cauchy sequences in S. Given some representative x^n of $\hat{x}^n$ show that there is another representative x'^n with specific asymptotic behavior. Using such representatives x'^n for all $n \in \mathbb{N}$ show that the equivalence class in $\overline{S}$ of the diagonal sequence $y := \{x'^n_n\}_{n\in\mathbb{N}}$ is a limit of $\hat{x}$.]
- Show that there is a natural isometric embedding (i.e., a map that preserves the metric) $i_S : S \to \overline{S}$. Furthermore, show that this is a bijection iff S is complete.

Definition 1.50. The metric space $\overline{S}$ constructed above is called the *completion* of the metric space S.

Proposition 1.51 (Universal property of completion)**.** *Let S be a metric space, T a complete metric space and $f : S \to T$ an isometric map. Then, there is a unique isometric map $\overline{f} : \overline{S} \to T$ such that $f = \overline{f} \circ i_S$. Furthermore, the closure of $f(S)$ in T is equal to $\overline{f}(\overline{S})$.*

Proof. **Exercise.** □

1.7 Norms and seminorms

In the following $\mathbb{K}$ will denote a field which can be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.52. Let V be a vector space over $\mathbb{K}$. Then a map $V \to \mathbb{R}^+_0 : x \mapsto \|x\|$ is called a *seminorm* iff it satisfies the following properties:

1. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}, x \in V$.
2. For all $x, y \in V : \|x + y\| \leq \|x\| + \|y\|$. (triangle inequality)

A seminorm is called a *norm* iff it satisfies in addition the following property:

3. $\|x\| = 0 \implies x = 0$.

Proposition 1.53. *Let V be a seminormed vector space over $\mathbb{K}$. Then, $d(v, w) := \|v - w\|$ defines a pseudometric on V. Moreover, d is a metric iff the seminorm is a norm.*

Proof. **Exercise.** □

Remark 1.54. Since a seminormed space is a pseudometric space all the concepts developed for pseudometric spaces apply. In particular the notions of convergence, Cauchy sequence and completeness apply to pseudonormed spaces.

Definition 1.55. A complete normed vector space is called a *Banach space*.

Exercise 7. Let S be a set and $\mathrm{F_b}(S, \mathbb{K})$ the set of bounded maps $S \to \mathbb{K}$.

1. $\mathrm{F_b}(S, \mathbb{K})$ is a vector space over $\mathbb{K}$.
2. The *supremum norm* on it is a norm defined by
$$\|f\|_{\sup} := \sup_{p \in S} |f(p)|.$$
3. $\mathrm{F_b}(S, \mathbb{K})$ with the supremum norm is a Banach space.

Exercise 8. Let S be a topological space and $\mathrm{C_b}(S, \mathbb{K})$ the set of bounded continuous maps $S \to \mathbb{K}$.

1. $\mathrm{C_b}(S, \mathbb{K})$ is a vector space over $\mathbb{K}$.
2. $\mathrm{C_b}(S, \mathbb{K})$ with the supremum norm is a Banach space.

Proposition 1.56. *Let V be a vector space with a seminorm $\|\cdot\|_V$. Consider the subset $A := \{v \in V : \|v\|_V = 0\}$. Then, A is a vector subspace. Moreover $v \sim w \iff v - w \in A$ defines an equivalence relation and $W := V/\sim$ is a vector space. The seminorm $\|\cdot\|_V$ induces a norm on W via $\|[v]\|_W := \|v\|_V$ for $v \in V$. Also, V is complete with respect to the seminorm $\|\cdot\|_V$ iff W is complete with respect to the norm $\|\cdot\|_W$.*

Proof. **Exercise**. □

Proposition 1.57. *Let V, W be seminormed vector spaces. Then, a linear map $\alpha : V \to W$ is continuous iff there exists a constant $c \geq 0$ such that*

$$\|\alpha(v)\|_W \leq c\|v\|_V \quad \forall v \in V.$$

Proof. **Exercise**. □